Analytical Solutions to the Navier-Stokes Equations with Density-dependent Viscosity and with Pressure

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Abstract

This article is the continued version of the analytical solutions for the pressureless Navier-Stokes equations with density-dependent viscosity [9]. We are able to extend the similar solutions structure to the case with pressure under some restriction to the constants γ and θ.

Key words: Navier-Stokes Equations, Analytical Solutions, Radial Symmetry, Density-dependent Viscosity, With Pressure

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1 Introduction

The Navier-Stokes equations can be formulated in the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \delta \nabla P = vis(\rho, u). \end{cases} \tag{1}$$

As usual, $\rho = \rho(x,t)$ and $u(x,t)$ are the density, the velocity respectively. $P = P(\rho)$ is the pressure. We use a γ-law on the pressure, i.e.

$$P(\rho) = K\rho^{\gamma}, \tag{2}$$

with $K > 0$, which is a universal hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. γ is the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma = 1$. It can be used for constructing models with non-degenerate isothermal fluid. δ can be the constant 0 or 1. When $\delta = 0$, we call the system is pressureless; when $\delta = 1$, we call that it is with pressure. And $vis(\rho, u)$ is the viscosity function. When $vis(\rho, u) = 0$, the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier-Stokes equations, see [1] and [3]. Here we consider the density-dependent viscosity function as follows:

$$vis(\rho, u) \doteq \bigtriangledown(\mu(\rho) \bigtriangledown \cdot u).$$

where $\mu(\rho)$ is a density-dependent viscosity function, which is usually written as $\mu(\rho) \doteq \kappa\rho^{\theta}$ with the constants κ, $\theta > 0$. For the study of this kind of the above system, the readers may refer [6], [8].

The Navier-Stokes equations with density-dependent viscosity in radial symmetry can be expressed by:

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) + \nabla K\rho^{\gamma} = (\kappa\rho^{\theta})_r\left(\frac{N-1}{r}u + u_r\right) + \kappa\rho^{\theta}(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u), \end{cases} \tag{3}$$

Recently, Yuen's results [9] showed that there exists a family of the analytical solutions for the pressureless Navier-Stokes equations with density-dependent viscosity:

for $\theta = 1$:

$$\begin{cases} \rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \; u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) = \frac{\lambda \dot{a}(t)}{a(t)^2}, a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2N\kappa}x^2 + \alpha, \end{cases} \tag{4}$$

for $\theta \neq 1$:

$$\begin{cases} \rho(t,r) = \begin{cases} \frac{y(r/a(t))}{a(t)^N}, & \text{for } y(\frac{r}{a(t)}) \geq 0; \\ 0, & \text{for } y(\frac{r}{a(t)}) < 0 \end{cases}, \; u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) = \frac{-\lambda \dot{a}(t)}{a(t)^{N\theta - N + 2}}, \; a(0) = a_0 > 0, \; \dot{a}(0) = a_1, \\ y(x) = \sqrt[\theta-1]{\frac{1}{2}(\theta - 1)\frac{-\lambda}{N\kappa\theta}x^2 + \alpha^{\theta-1}}, \end{cases} \tag{5}$$

where $\alpha > 0$.

In this article, we extend the results form the study of the analytical solutions in the Navier-Stokes equations without pressure [9] to the case with pressure. The techniques of separation method of self-similar solutions were also found to treat other similar systems in [2], [4], [5], [7], [8] and [9].

Our main result is the following theorem:

Theorem 1 *For the N-dimensional Navier-Stokes equations in radial symmetry (3), there exists a family of solutions, those are:*

for $\theta = \gamma = 1$,

$$\begin{cases} \rho(t,r) = \frac{Ae^{B(\frac{r}{a(t)})^2 + C}}{a(t)^N}, \; u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa \dot{a}(t)}{a(t)^2} = 0, \; a(0) = a_0 > 0, \; \dot{a}(0) = a_1, \end{cases} \tag{6}$$

where $A \geq 0$, B and C are constants.

for $\theta = \gamma > 1$,

$$\begin{cases} \rho(t,r) = \begin{cases} \frac{y(\frac{r}{a(t)})}{a(t)^N}, \; for \; y(\frac{r}{a(t)}) \geq 0 \\ 0, \; for \; y(\frac{r}{a(t)}) < 0 \end{cases}, \; u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \frac{\ddot{a}(t)}{a(t)^N} + \frac{K\gamma}{a(t)^{\theta N+1}} - \frac{N\kappa\theta \dot{a}(t)}{a(t)^{\theta N+2}} = 0, \; a(0) = a_0 > 0, \; \dot{a}(0) = a_1, \\ y(z) = \sqrt[\theta-1]{\frac{1}{2}(\theta - 1)z^2 + \alpha^{\theta-1}}, \end{cases} \tag{7}$$

where a_0, a_1 and $\alpha > 0$ are constants;

for $\frac{\gamma}{2}+\frac{1}{2}-\frac{1}{N}=\theta\geq 1-\frac{1}{N}$,

$$\begin{cases} \rho(t,r)=\begin{cases}\frac{y(\frac{r}{a(t)})}{a(t)^N},\ for\ y(\frac{r}{a(t)})\geq 0\\ 0,\ for\ y(\frac{r}{a(t)})<0\end{cases},\ u(t,r)=\frac{\dot{a}(t)}{a(t)}r \\ a(t)=\sigma(mt+n)^s,\ 0<s=\frac{2}{\gamma N-N+2}\leq 1 \\ \left[\frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2}-\frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\right]\dot{y}(z)=\frac{(1-s)m^2}{\sigma^{N-1}}z,\ y(0)=\alpha>0 \end{cases} \tag{8}$$

where $m,\ n>0,\ \sigma>0$ *and* α *are constants.*

2 Separation Method of Self-Similar Solutions

Before we present the proof of Theorem 1, Lemmas 3 and 12 of [9] are quoted here.

Lemma 2 (Lemma 3 of [9]) *For the equation of conservation of mass in radial symmetry:*

$$\rho_t+u\rho_r+\rho u_r+\frac{N-1}{r}\rho u=0, \tag{9}$$

there exist solutions,

$$\rho(t,r)=\frac{f(r/a(t))}{a(t)^N},\ u(t,r)=\frac{\dot{a}(t)}{a(t)}r, \tag{10}$$

with the form $f\geq 0\in C^1$ *and* $a(t)>0\in C^1$.

Lemma 3 (Lemma 12 of [9]) *For the ordinary differential equation*

$$\begin{cases} \dot{y}(z)y(z)^n-\xi x=0, \\ y(0)=\alpha>0, n\neq -1, \end{cases} \tag{11}$$

where ξ *and* n *are constants,*

we have the solution

$$y(z)=\sqrt[n+1]{\frac{1}{2}(n+1)\xi z^2+\alpha^{n+1}}, \tag{12}$$

where the constant $\alpha>0$.

At this stage, we can show the proof of Theorem 1.

Proof. Our solutions (6), (7) and (8) fit the mass equation $(3)_1$ by Lemma (2). Next, for the equation $(3)_2$, we plug our solutions to check that.

For $\theta = \gamma = 1$, we get

$$\rho(u_t + u \cdot u_r) + K\left[\rho\right]_r - (\kappa\rho)_r\left(\frac{N-1}{r}u + u_r\right) - \kappa\rho_r(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \tag{13}$$

$$= \frac{Ae^{B(\frac{r}{a(t)})^2+C}}{a(t)^3}\frac{\ddot{a}(t)}{a(t)}r + K\frac{Ae^{B(\frac{r}{a(t)})^2+C}}{a(t)^4}B\left[\frac{-2r}{a(t)}\right] - \frac{AN\kappa e^{B(\frac{r}{a(t)})^2+C}}{a(t)^4}B\left[\frac{-2r}{a(t)}\right]\frac{\dot{a}(t)}{a(t)} \tag{14}$$

$$= \frac{Ae^{B(\frac{r}{a(t)})^2+C}r}{a(t)^4}\left[\ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa\dot{a}(t)}{a(t)^2}\right] \tag{15}$$

$$= 0, \tag{16}$$

where the function $a(t)$ is required to be

$$\ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa\dot{a}(t)}{a(t)^2} = 0,\ a(0) = a_0 > 0,\ \dot{a}(0) = a_1, \tag{17}$$

where a_0 and a_1 are constants.

For $\theta = \gamma > 1$, we have:

$$\rho(u_t + u \cdot u_r) + K\left[\rho^\gamma\right]_r - (\kappa\rho^\theta)_r\left(\frac{N-1}{r}u + u_r\right) - \kappa\rho_r^\theta(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \tag{18}$$

$$= \frac{y(\frac{r}{a(t)})}{a(t)^N}\frac{\ddot{a}(t)}{a(t)}r + \frac{K\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{a(t)^{\gamma N+1}} - \frac{N\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{a(t)^{\theta N+2}}\dot{a}(t). \tag{19}$$

By defining $z := r/a(t)$, and requiring

$$y(z)z - y(z)^{\theta-1}\dot{y}(z) = 0, \tag{20}$$

$$z - y(z)^{\theta-2}\dot{y}(z) = 0, \tag{21}$$

(19) becomes

$$= y(z)z\left[\frac{\ddot{a}(t)}{a(t)^N} + \frac{K\theta}{a(t)^{\theta N+1}} - \frac{N\kappa\theta\dot{a}(t)}{a(t)^{\theta N+2}}\right] = 0, \tag{22}$$

where the function $a(t)$ is required to be

$$\frac{\ddot{a}(t)}{a(t)^N} + \frac{K\gamma}{a(t)^{\theta N+1}} - \frac{N\kappa\theta\dot{a}(t)}{a(t)^{\theta N+2}} = 0. \tag{23}$$

Therefore, the equation $(3)_2$ is satisfied.

With $n := \theta - 2$ and $\xi := 1$, in Lemma 3, (21) can be solved by

$$y(z) = \sqrt[\theta-1]{\frac{1}{2}(\theta-1)z^2 + \alpha^{\theta-1}}, \tag{24}$$

where $\alpha > 0$ is a constant.

For the case of $\frac{\gamma}{2} + \frac{1}{2} - \frac{1}{N} = \theta \geq 1 - \frac{1}{N}$, we have,

$$\rho(u_t + u \cdot u_r) + K\left[\rho^{\gamma}\right]_r - (\kappa\rho^{\gamma})_r\left(\frac{N-1}{r}u + u_r\right) - \kappa\rho_r^{\theta}(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \tag{25}$$

$$= \frac{y(\frac{r}{a(t)})}{a(t)^N}\frac{\ddot{a}(t)}{a(t)}r + \frac{K\theta y(\frac{r}{a(t)})^{\gamma-1}\dot{y}(\frac{r}{a(t)})}{a(t)^{\gamma N+1}} - \frac{N\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{a(t)^{\theta N+2}}\dot{a}(t). \tag{26}$$

By letting $a(t) = \sigma(mt+n)^s$, we have

$$= y(\frac{r}{a(t)})\frac{s(s-1)(mt+n)^{s-2}}{\sigma^N(mt+n)^{sN}}\frac{m^2\sigma r}{a(t)} + \frac{K\theta y(\frac{r}{a(t)})^{\gamma-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}} - \frac{sm\sigma N\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\theta N+2}(mt+n)^{s(\theta N+2)}}(mt+n)^{s-1} \tag{27}$$

$$= y(\frac{r}{a(t)})\frac{s(s-1)m^2 r}{\sigma^{N-1}a(t)}\frac{1}{(mt+n)^{sN-(s-2)}} + \frac{K\theta y(\frac{r}{a(t)})^{\gamma-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}} - \frac{smN\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\theta N+1}(mt+n)^{s(\theta N+2)-(s-1)}} \tag{28}$$

$$= y(\frac{r}{a(t)})\frac{s(s-1)m^2 r}{\sigma^{N-1}a(t)}\frac{1}{(mt+n)^{sN-s+2}} + \frac{K\theta y(\frac{r}{a(t)})^{\gamma-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}} - \frac{smN\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{\sigma^{\theta N+1}(mt+n)^{\theta N+s+1}}. \tag{29}$$

Here we require that

$$\begin{cases} sN - s + 2 = s(\gamma N+1), \\ s(\gamma N+1) = s(\theta N+1) + 1. \end{cases} \tag{30}$$

That is

$$0 < s = \frac{1}{(\gamma-\theta)N} = \frac{2}{\gamma N - N + 2} \leq 1. \tag{31}$$

In the solutions (8), it fits to the conditions (31) by setting $\frac{\gamma}{2} + \frac{1}{2} - \frac{1}{N} = \theta \geq 1 - \frac{1}{N} > 0$ and $s = \frac{2}{\gamma N - N + 2}$. Additionally by defining $z := r/a(t)$, the equation (29) becomes

$$= \frac{y(z)s}{(mt+n)^{Ns-s+2}}\left[\frac{(s-1)m^2}{\sigma^{N-1}}z + \frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2}\dot{y}(z) - \frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\dot{y}(z)\right], \tag{32}$$

Here we require that

$$\left[\frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2} - \frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\right]\dot{y}(z) = \frac{(1-s)m^2}{\sigma^{N-1}}z. \tag{33}$$

The proof is completed. ■

In the corollary can be followed immediately:

Corollary 4 *For $m < 0$, the solutions (8) blowup at the finite time $T = -m/n$.*

References

[1] G.Q. Chen and D.H. Wang, *The Cauchy Problem for the Euler Equations for Compressible Fluids*, Handbook of Mathematical Fluid Dynamics, Vol. I, 421-543, North-Holland, Amsterdam, 2002.

[2] Y.B. Deng, J.L. Xiang and T. Yang, *Blowup Phenomena of Solutions to Euler-Poisson Equations*, J. Math. Anal. Appl. **286** (1)(2003), 295-306.

[3] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vols. 1, 2, 1998, Oxford:Clarendon Press, 1998.

[4] T.H. Li, *Some Special Solutions of the Multidimensional Euler Equations in R^N*, Comm. Pure Appl. Anal. **4** (4)(2005), 757-762.

[5] T. Makino, *Blowing up Solutions of the Euler-Poission Equation for the Evolution of the Gaseous Stars*, Transport Theory and Statistical Physics **21** (1992), 615-624.

[6] T. Nishida, *Equations of fluid dynamics-free surface problems*, Comm. Pure Appl. Math., XXXIX (1986), 221–238.

[7] M.W. Yuen, *Blowup Solutions for a Class of Fluid Dynamical Equations in R^N*, J. Math. Anal. Appl. **329** (2)(2007), 1064-1079.

[8] M.W. Yuen, *Analytical Blowup Solutions to the 2-dimensional Isothermal Euler-Poisson Equations of Gaseous Stars*, J. Math. Anal. Appl. **341 (1)(2008),** 445-456.

[9] M.W. Yuen, *Analyitcal Solutions to the Navier-Stokes Equations*, J. Math. Phys., **49** (2008) No. 11, 113102, 10pp.